HOMEWORK 1

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1. Find all the solutions to

$$z^2 = -5 - 12i.$$

Hint: Use the Cartesian representation of complex numbers and the list of Pythagorean triples.

2. Sketch the set

$$\left\{ z \in \mathbb{C} : \left| \frac{z-i}{z+i} \right| \le \frac{1}{2} \right\}$$

3. Find the image of $D = \{z \in \mathbb{C} : \operatorname{Re} z < 0 \text{ and } |\operatorname{Im} z| \le \pi\}$ under the map $f(z) = e^z$. Also find the image of $G = \{z \in \mathbb{C} : \operatorname{Re} z > 0 \text{ and } \operatorname{Im} z > 0\}$ under $f(z) = z^2$.

4. Show z_1, z_2, z_3 form an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

5. For $n = 1, 2, \dots$, find an explicit formula in terms of n for:

$$1 + \cos\left(\frac{2\pi}{n}\right) + \cos\left(2\frac{2\pi}{n}\right) + \cdots + \cos\left((n-1)\frac{2\pi}{n}\right).$$

6. Prove the parallelogram inequality:

$$|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2).$$

7. Find the set of z's where $\sum_{n=1}^{\infty} n z^n$ converges.

8*. Suppose that $f : [a,b] \to \mathbb{C}$ is continuous. Let the average of f over the interval $I = [a,b]$ be

$$A := \frac{1}{b-a} \int_a^b f(x)dx = \fint_I f.$$

Show that $|f(x)| \le |A|$ for all $x \in [a,b]$ implies that f is constant.

Hint: First show that $|A| = \fint_I |f|$, and thus $|f|$ is constant and equals $|A|$. Therefore, we can write $f(x) = |A|e^{i\theta(x)}$ $(x \in I)$ and $A = |A|e^{i\alpha}$ for some $\theta(x), \alpha \in [0, 2\pi)$. Finally, show that $\theta(x)$ is constant on I and equals α.

9*. Stereographic projection sends a point $z \in \mathbb{C}$ to a point $z^* \in S^2 \setminus \{N\}$ one-to-one and onto:

$$z^* = \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right).$$

Conversely, if $w^* = (x_1, x_2, x_3)$, then

$$w = \frac{x_1}{1-x_3} + i\frac{x_2}{1-x_3}.$$

Rigid motions of the unit sphere can be used to describe some transformations of the plane.
(a) Map $z \in \mathbb{C}$ to S^2, apply a rotation of the unit sphere about the x_3 axis, then map the resulting point back to the plane. Do the same for the x_1 axis.

Another map can be obtained by mapping z to S^2, then translating the sphere so that the origin is sent to (x_0, y_0, z_0), then projecting back to the plane. The projection to the plane is given by drawing a line through the (translated) north pole and a point on the (translated) sphere and finding the intersection with the plane $\{(x, y, 0)\}$.
(b) Find this map as an explicit function of z int he case of a fixed vertical translation, i.e., when $(0, 0, 0)$ is sent to $(0, 0, h)$.

Partial answer: the maps in part (a) and (b) are of the form $(az + b)/(cz + d)$ with $ad - bc \neq 0$

For a movie of these maps, see http://www.youtube.com/watch?v=JX3VmDgiFnY

10*. Formally solve the cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in \mathbb{C}$, $a \neq 0$, by the following reduction process:

(a) Set $x = u + t$ and choose the constant t so that the coefficient of u^2 is equal to zero.
(b) If the coefficient of u is also zero, then take a cube root to solve. If the coefficient of u is non-zero, set $u = kv$ and choose the constants k and a so that $v^3 = 3v + r$, for some constant r.
(c) Set $v = z + 1/z$ and obtain a quadratic equation for z^3. The map $z + 1/z$ is important for several reasons and is known as the Joukowsky map.
(d) Use the quadratic formula to find two possible values for z^3, and then take a cube root to solve for z.

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